Proof by Mathematical Induction

SCIS 313: Data Structures and Algorithm Analysis

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The Induction Method

Mathematical induction is a powerful proof technique for statements involving natural numbers. Think of it like dominoes: if you can prove the first domino falls (base case) and that any falling domino causes the next one to fall (inductive step), then all dominoes must fall.

Three-Step Template

Step 1: Base Case

- Pick the smallest value (usually $n = 1$ or $n = 0$)
- Verify that the statement $P(n)$ is true for this value
- Show your work: substitute the base value into both sides of the equation

Step 2: Inductive Hypothesis

- **Assume** the statement $P(k)$ is true for some arbitrary $k \geq$ (base case)
- Write out what this assumption means explicitly
- You're NOT proving this, you're assuming it to prove the next step

Step 3: Inductive Step

- **Prove** that $P(k+1)$ is true, using the assumption from Step 2
- Start with the left-hand side (LHS) of $P(k+1)$
- Manipulate it algebraically until you can use the inductive hypothesis
- Show that LHS = RHS (right-hand side)

Conclusion: By mathematical induction, $P(n)$ is true for all $n \geq$ (base case).

Problems

Problem 1: Sum of First n Natural Numbers (Warm-up)

Prove: For all $n \geq 1$,

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Hint

Step 1: Try $n = 1$. Does $1 = \frac{1(1+1)}{2}$?
Step 3: When proving $P(k+1)$, you'll need to add $(k+1)$ to both sides of $P(k)$.
Split the sum: $\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{k} i\right) + (k+1)$
Then use the inductive hypothesis to replace $\sum_{i=1}^{k} i$.

Problem 2: Sum of First n Odd Numbers

Prove: For all $n \geq 1$,

$$\sum_{i=1}^{n}(2i-1) = 1 + 3 + 5 + \cdots + (2n-1) = n^2$$

Hint

Base case: What is the first odd number? Does it equal 1^2?
Step 3: The $(k+1)$-th odd number is $2(k+1) - 1 = 2k + 1$.
Split: $\sum_{i=1}^{k+1}(2i-1) = \left(\sum_{i=1}^{k}(2i-1)\right) + (2k+1)$
After using the inductive hypothesis, factor to get $(k+1)^2$.

Problem 3: Sum of Squares

Prove: For all $n \geq 1$,

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Hint

Base case: Does $1^2 = \frac{1 \cdot 2 \cdot 3}{6}$?
Step 3: You'll need to add $(k+1)^2$ to both sides.
After using the inductive hypothesis:

$$\frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

Factor out $(k+1)$ and find a common denominator. You should get:

$$\frac{(k+1)(k+2)(2k+3)}{6} = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

Problem 4: Geometric Series

Prove: For all $n \geq 0$,

$$\sum_{i=0}^{n} 2^i = 1 + 2 + 4 + 8 + \cdots + 2^n = 2^{n+1} - 1$$

Hint

Base case: Start with $n = 0$. Does $2^0 = 2^{0+1} - 1$?
Step 3: Add 2^{k+1} to both sides of $P(k)$.

$$\left(\sum_{i=0}^{k} 2^i\right) + 2^{k+1} = (2^{k+1} - 1) + 2^{k+1}$$

Simplify the right side: $2 \cdot 2^{k+1} - 1 = 2^{k+2} - 1 = 2^{(k+1)+1} - 1$ ✓

Problem 5: Inequality - Powers of 2

Prove: For all $n \geq 4$,

$$2^n > n^2$$

Hint

Base case: Use $n = 4$. Is $2^4 > 4^2$? (i.e., is $16 > 16$?)
Actually, $2^4 = 16 = 4^2$, so we need $n = 5$ as our base case.
Verify: $2^5 = 32 > 25 = 5^2$ ✓
Step 2: Assume $2^k > k^2$ for some $k \geq 5$.
Step 3: We need to show $2^{k+1} > (k+1)^2$.
Start with: $2^{k+1} = 2 \cdot 2^k > 2 \cdot k^2$ (using the inductive hypothesis)
Now show that $2k^2 > (k+1)^2$ for $k \geq 5$:

$$2k^2 > (k+1)^2$$
$$2k^2 > k^2 + 2k + 1$$
$$k^2 > 2k + 1$$
$$k^2 - 2k - 1 > 0$$

This is true for $k \geq 5$ (you can verify by plugging in $k = 5$).

Problem 6: Divisibility

Prove: For all $n \geq 1$, the expression $n^3 - n$ is divisible by 3.
Equivalently, prove: $3 \mid (n^3 - n)$ for all $n \geq 1$.

Hint

Base case: $n = 1$: Is $1^3 - 1 = 0$ divisible by 3? Yes! ✓
Step 2: Assume $k^3 - k = 3m$ for some integer m (i.e., $k^3 - k$ is divisible by 3).
Step 3: Show $(k+1)^3 - (k+1)$ is divisible by 3.
Expand $(k+1)^3$:

$$\begin{aligned}(k+1)^3 - (k+1) &= (k^3 + 3k^2 + 3k + 1) - (k+1) \\ &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= k^3 - k + 3k^2 + 3k \\ &= (k^3 - k) + 3(k^2 + k)\end{aligned}$$

The first term is divisible by 3 (inductive hypothesis), and the second term is clearly divisible by 3.

Problem 7: Triangular Loop Complexity

Consider this code:

```
int count = 0;
for (int i = 1; i <= n; i++) {
    for (int j = 1; j <= i; j++) {
        count++;
    }
}
```

Prove: After the loops complete, `count` $= \frac{n(n+1)}{2}$ for all $n \geq 1$.

Hint

Think of this as proving the sum formula, but in terms of code execution.
Step 1: Base case $n = 1$:

- Outer loop runs once ($i = 1$)
- Inner loop runs once ($j = 1$)
- `count` $= 1$
- Formula: $\frac{1(1+1)}{2} = 1$ ✓

Step 2: Assume after k iterations of the outer loop, `count` $= \frac{k(k+1)}{2}$.
Step 3: When the outer loop runs for $i = k + 1$:

- Previous count was $\frac{k(k+1)}{2}$
- Inner loop adds $(k + 1)$ to count
- New count $= \frac{k(k+1)}{2} + (k + 1)$
- Factor: $(k + 1)\left(\frac{k}{2} + 1\right) = (k + 1)\left(\frac{k+2}{2}\right) = \frac{(k+1)(k+2)}{2}$ ✓

Problem 8: Binary Search Depth (Challenge)

Consider a binary search tree with n nodes arranged in a complete binary tree.

Prove: The maximum depth (height) of the tree is $\lfloor \log_2 n \rfloor + 1$ for all $n \geq 1$.

Equivalently, prove: A complete binary tree of height h contains at most $2^h - 1$ nodes.

Hint

This is trickier—prove the equivalent statement about nodes.
Base case: $h = 1$ (just a root): Does it have at most $2^1 - 1 = 1$ nodes? Yes! ✓
Step 2: Assume a complete binary tree of height k has at most $2^k - 1$ nodes.
Step 3: A tree of height $k + 1$ consists of:

- A root node (1 node)
- A left subtree of height at most k (at most $2^k - 1$ nodes by hypothesis)
- A right subtree of height at most k (at most $2^k - 1$ nodes by hypothesis)

Total nodes: $1 + (2^k - 1) + (2^k - 1) = 1 + 2 \cdot 2^k - 2 = 2^{k+1} - 1$ ✓
Connection to logarithm: If a tree has n nodes and height h, then:

$$n \leq 2^h - 1 \implies n + 1 \leq 2^h \implies \log_2(n + 1) \leq h$$

Additional Tips for Success

Common Mistakes to Avoid

1. **Don't prove the base case using the formula.** You must verify it directly.
2. **Don't assume what you're trying to prove.** In Step 3, you cannot assume $P(k+1)$ is true—you must derive it.
3. **State the inductive hypothesis clearly.** Write "Assume $P(k)$ is true" and write out what that means.
4. **Show all algebraic steps.** Don't skip steps in your manipulation—this is where most errors occur.
5. **End with a clear conclusion.** "Therefore, by mathematical induction, $P(n)$ is true for all $n \geq$ [base case]."

Useful Algebraic Identities

These come up frequently in induction proofs:

$$\begin{aligned}(k+1)^2 &= k^2 + 2k + 1\\ (k+1)^3 &= k^3 + 3k^2 + 3k + 1\\ 2^{k+1} &= 2 \cdot 2^k\\ (k+1)! &= (k+1) \cdot k!\end{aligned}$$

Factoring:

$$\begin{aligned}k^2 + 3k + 2 &= (k+1)(k+2)\\ k^2 + k &= k(k+1)\\ 2k^2 + 3k + 1 &= (k+1)(2k+1)\end{aligned}$$

General Strategy

1. **Understand the pattern.** Before starting, verify the formula for $n = 1, 2, 3$ to build intuition.
2. **Write $P(k)$ and $P(k+1)$ explicitly.** This helps you see what you need to prove.
3. **In Step 3, always start with the LHS of $P(k+1)$.** Manipulate it to match the RHS.
4. **Look for ways to "peel off" the $(k+1)$ term.** Often you can write:

$$\text{Sum from 1 to } k+1 = (\text{Sum from 1 to } k) + (\text{the } k+1 \text{ term})$$

5. **Use the inductive hypothesis exactly once.** Replace the $P(k)$ portion with its formula.
6. **Factor aggressively.** The goal is to get the expression into the form of $P(k+1)$.

Submission Guidelines

For each problem, your proof should include:

- **Step 1 (Base Case):** Clear verification with all calculations shown
- **Step 2 (Inductive Hypothesis):** Explicit statement of the assumption
- **Step 3 (Inductive Step):** Complete algebraic derivation from $P(k)$ to $P(k+1)$
- **Conclusion:** "By mathematical induction, $P(n)$ is true for all $n \geq$ ..."